Recent progress on graphs forbidding even holes

Dewi Sintiari

Wilfrid Laurier University

Ontario Combinatorics Workshop

Ottawa, Canada

12-13 May 2023

Induced subgraph

H is an *induced subgraph* of graph G, if it can be obtained by **deleting some vertices** of G.

- Deleting a vertex v from G means:
  delete v + all edges incident to v

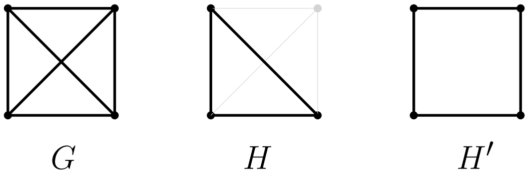
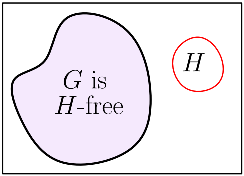


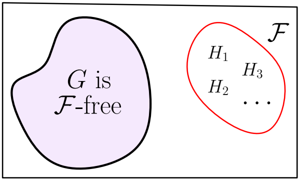
Figure: H is an induced subgraph of G, but H' is not

Forbidden induced subgraph

If G does not contain H as an induced subgraph, then we say:

- G is *H-free*
- G *forbids* H
- G *excludes* H
- H is a *forbidden induced subgraph* of G





A class *$\mathcal{C}$ is $\mathcal{F}$-free*, if $\forall G \in \mathcal{C}$, G is $\mathcal{F}$-free

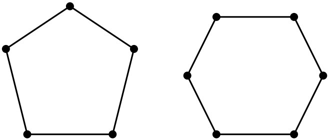
Why studying classes of graphs forbidding certain induced subgraphs?

- ~~Many choices of forbidden induced subgraphs, so this could lead to publishing many papers.~~
- Many graph problems that are NP-hard (NP-complete) can be tackled when some structures are forbidden.
- Many graph classes have nice characterizations when some structures are forbidden.
- Induced subgraphs appear naturally in real-world (deleting vertex = deleting object).

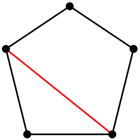
Topics to explore in this area of research:

- Structural properties;
- Algorithms for combinatorial graph problems;
- many others...

Forbidding *holes*



chordless cycles (holes)



cycle with chord

A *hole* H in a graph is a chordless cycle of length at least 4.

It is *even* or *odd* depending on the parity of $|V(H)|$.

Well-known graph classes forbidding holes

- *Chordal graphs*: **forbid <u>all holes</u>**.

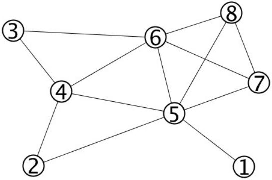


Figure: An example: a chordal graph with eight vertices

- *Perfect graphs:* **forbid <u>all odd holes</u>** in the graph and in its complement.
- *Even-hole-free graphs:* **forbid <u>all even holes</u>** (and implicitly, all even holes of length ≥ 6 in its complement).

Four configurations that appear when forbidding holes*

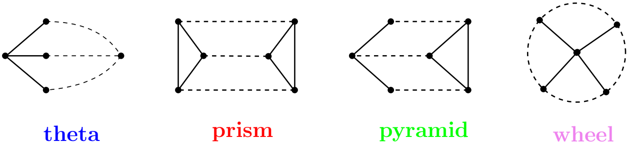


Figure: **4-configurations** (*dashed lines represent paths of length at least 1*)

- No odd holes ⇒ no (pyramid, $\text{wheel}_{\text{type-1}}$)-free.
- No even holes ⇒ no (theta, prism, $\text{wheel}_{\text{type-2}}$)-free.

---

*The 4-configurations are actually called *Truemper configurations*; there is a nice survey about this written by Kristina Vušković.

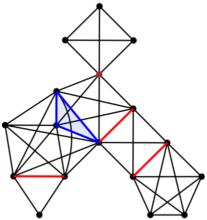
Four configurations that appear when forbidding holes

If G is **hole-free**, then:

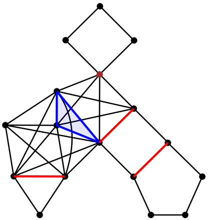
- G is a clique; or
- G has a clique cutset
[†]

If G is **4-configurations-free**, then:

- G is a clique or G is a hole; or
- G has a clique cutset



Graphs forbidding ***all holes***



Graphs forbidding ***4-configurations***

Figure: Tree-like structure of *chordal graphs* and *4-configuration-free graphs*

[†] a *clique* in a graph is a set of pairwise adjacent vertices; a *cutset* of G is a set of vertices whose removal increases # connected components of G; a *clique cutset* is a cutset that induces a clique

Why studying:

even-hole-free (ehf) graphs

Motivation behind the study of even-hole-free graphs

Comparison between <u>even-hole-free graphs</u> & <u>perfect graphs</u>

	Ehf graphs	Perfect graphs
Structure	“simpler”	more complex
Maximum clique	polynomial	polynomial
Coloring	?	polynomial
Maximum independent set	?	polynomial

Schema of decomposition theorem

If G belongs to $\mathcal{C}$ then either G is "basic (simple)" or G has some "cutset with some particular property".

A *vertex cutset* (*edge cutset*) of a connected graph G is a set of vertices (edges) S such that $G \setminus S$ is not a connected graph.

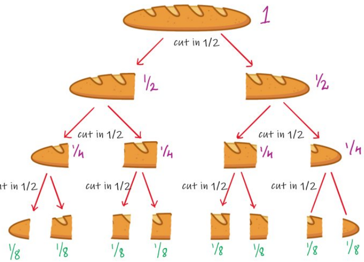


Figure: Illustration of graph decomposition

2nd approach:

forbidding more induced subgraphs

A tool: width parameters

Treewidth (tw):

a parameter measuring how similar a graph from being a tree

A chordalization of G is a graph obtained by adding edges to G to make it chordal

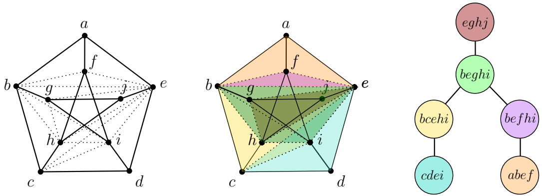


Figure: A chordalization of a graph and its *tree decomposition*

Courcelle's theorem: many graph optimization problems are polynomially solvable when the treewidth is small.

Other width parameters: *rankwidth*, *cliquewidth*, *branchwidth*, etc., they bound one to each other.

Studied by *Silva, da Silva, Sales (2010)*

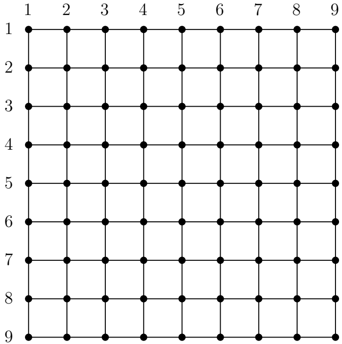


Figure: 9×9 grid

Studied by *Silva, da Silva, Sales (2010)*

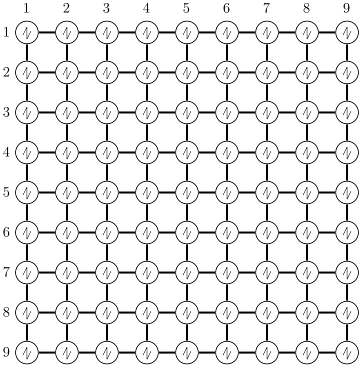


Figure: 9×9 grid-minor

Planar even-hole-free graphs

Studied by *Silva, da Silva, Sales (2010)*

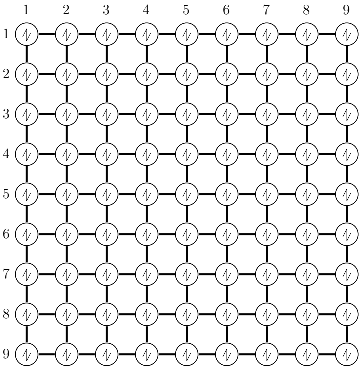
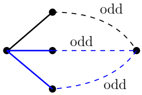
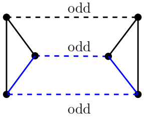


Figure: 9×9 grid-minor

If a planar ehf graph G contains a 9×9 grid-minor, then it contains either:



theta



prism

Theorem 1

If G is planar ehf, then $tw(G) \leq 49$.

Triangle-free even-hole-free graphs

Studied by *Conforti, Cornuéjols, Kapoor, Vušković (2000), and Cameron, da Silva, Huang, Vušković (2016)*

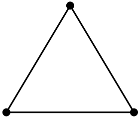


Figure: *Triangle*

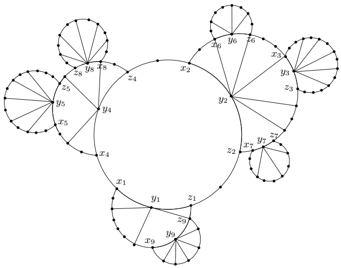


Figure: Every triangle-free ehf graph can be formed this way

Studied by *Conforti, Cornuéjols, Kapoor, Vušković (2000), and Cameron, da Silva, Huang, Vušković (2016)*

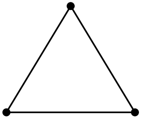


Figure: *Triangle*

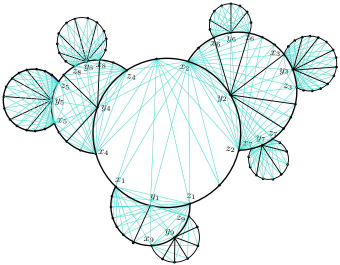


Figure: A chordalization triangle-free ehf graph with clique number 6

Theorem 2

If G is triangle-free ehf, then $tw(G) \leq 5$.

Studied by *Adler, Le, Müller, Radovanović, Trotignon, Vušković (2017)*

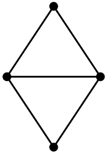


Figure: *Diamond*

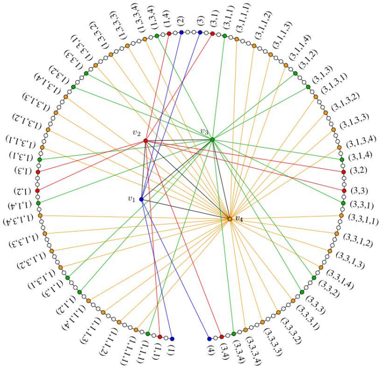


Figure: A family of diamond-free ehf graphs with high rankwidth

Studied by *S. & Trotignon (2021))*

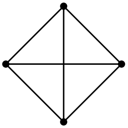


Figure: K_4 (i.e., 4-vertex complete graph)

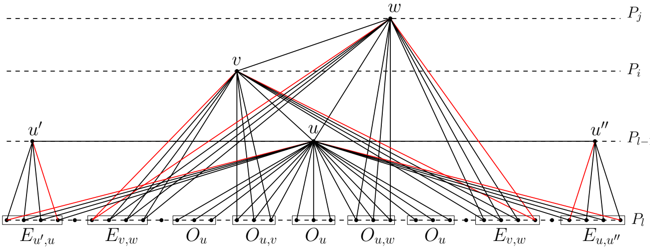


Figure: *Layered wheel*, a family of K_4-free ehf graphs with arbitrarily large treewidth

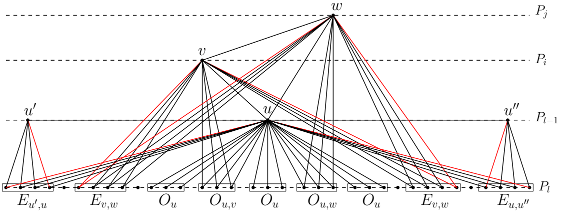
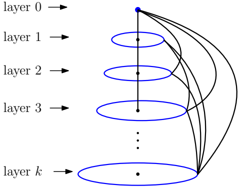
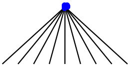


Figure: *Layered wheel*

Observation: the graphs contain a **big clique minor** and **vertices of high degree**.





When:

the size of the clique minor is bounded

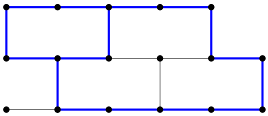
Studied by *Aboulker, Adler, Kim, S., Trotignon (2021)*

Theorem 3

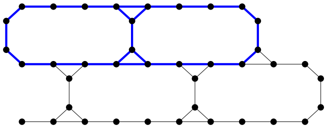
If G is:

- *K_t-minor-free*
- *has treewidth $\geq f_{K_t}(k)$*

Then G contains one of the following as an induced subgraph.



$k \times k$-wall



line graph of $k \times k$-wall

When:

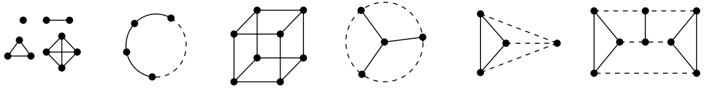
the maximum degree is bounded

Studied by *Aboulker, Adler, Kim, S., Trotignon (2021)*

Even-hole-free graphs of maximum degree *at most 3*

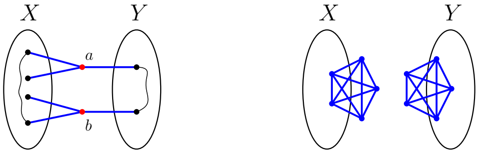
Structure theorems:

Basic graphs:



K_n, $n \leq 4$ | hole | cube | proper wheel | pyramid | extended prism

Cutsets:



Implication: this class has *treewidth* ≤ 3.

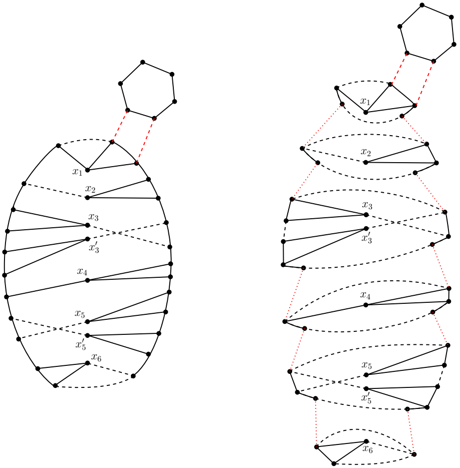
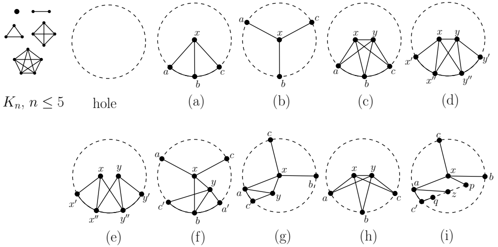


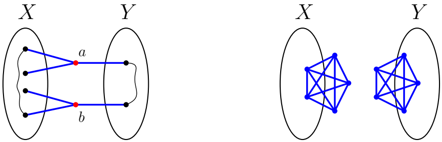
Figure: Decomposition of a non-basic subcubic even-hole-free graphs

Even-hole-free graphs of maximum degree *at most 4*

Basic graphs:



Cutsets:



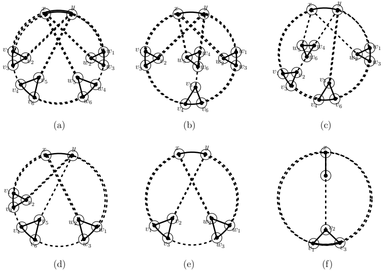
Implication: this class has *treewidth* ≤ 4.

Even-hole-free graphs of maximum degree $d \geq 5$

Conjecture 1

Let G be a even-hole-free graph with maximum degree d, then one of the following holds.

1. *G is a basic graph;*
2. *G has a clique cutset.*



Even-hole-free graphs of maximum degree $d \geq 5$

Conjecture 1

Let G be a even-hole-free graph with maximum degree d, then one of the following holds.

1. *G is a basic graph;*
2. *G has a clique cutset.*

Result by Abrishami, Chudnovsky, Vušković (2020):

> *Even-hole-free graphs of maximum degree d have treewidth bounded by some function of d.*

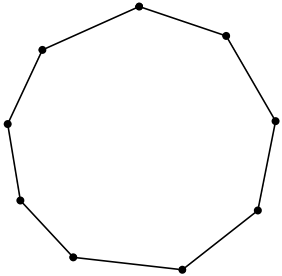
Consequence: for even-hole-free graphs of maximum degree d, many combinatorial graph problems are solvable in polynomial time.

A class related to ehf:

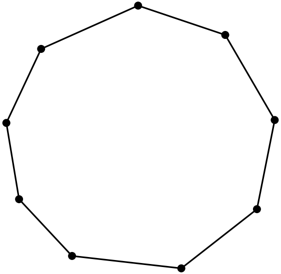
graphs whose all holes have same *odd* length (≥ 7)

Studied by *Horsfield, Preissmann, Robin, S., Trotignon, Vušković (2022)*

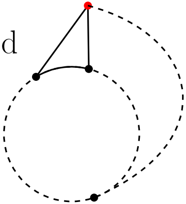
Graphs whose all holes have same *odd* length (1st case)



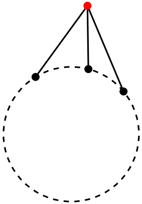
Graphs whose all holes have same *odd* length (1st case)



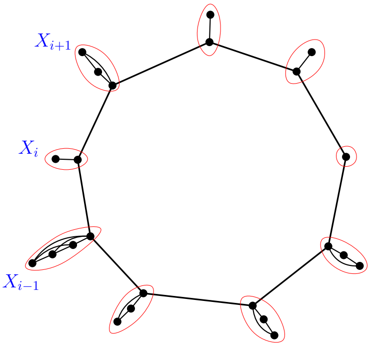
pyramid



wheel



Graphs whose all holes have same *odd* length (1st case)



Graphs whose all holes have same *odd* length (1st case)

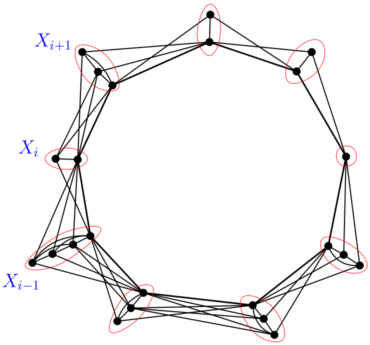
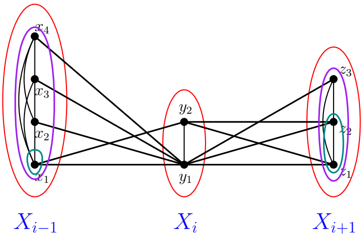
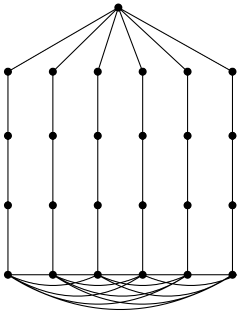


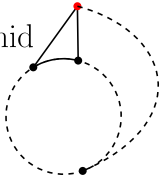
Figure: The first basic graph (*ring*)

Graphs whose all holes have same *odd* length (1st case)

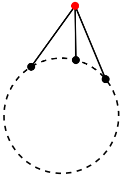


Graphs whose all holes have same *odd* length (2nd case)



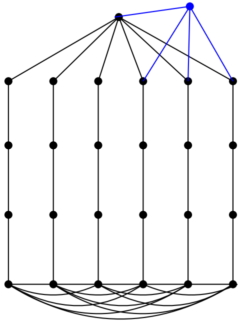


pyramid

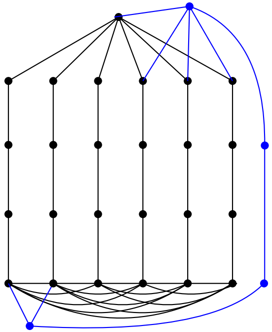


wheel

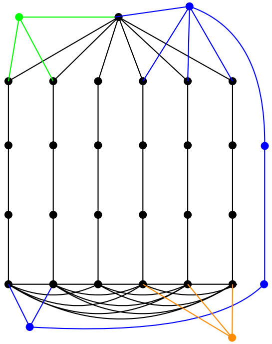
Graphs whose all holes have same *odd* length (2nd case)



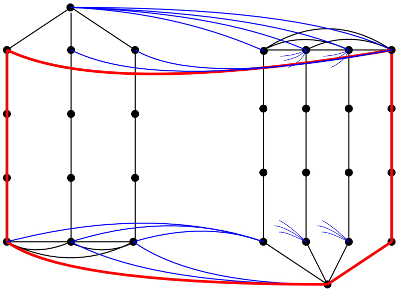
Graphs whose all holes have same *odd* length (2nd case)



Graphs whose all holes have same *odd* length (2nd case)

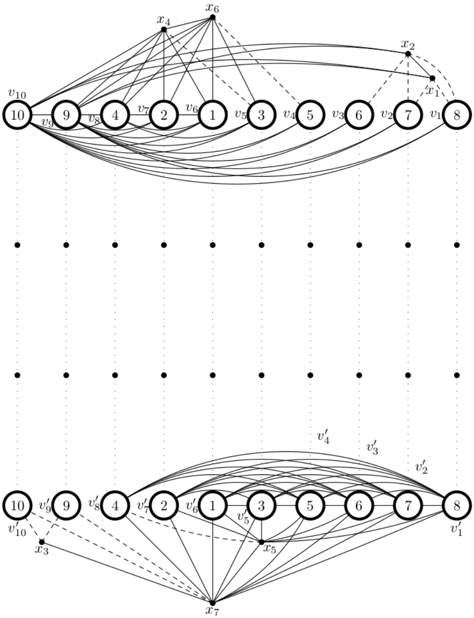


Graphs whose all holes have same *odd* length (2nd case)



hole of length 9

Graphs whose all holes have same *odd* length (2nd case)



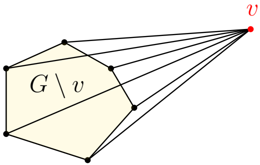
Theorem: Structure theorem

Let $\mathcal{C}_k$ be the class of graphs whose holes are all of odd length k, $k \geq 7$. Then for every graph $G \in \mathcal{C}_k$, one of the following holds:

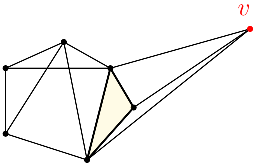
- G is a *ring* of length k; or
- G is a "blowup" of an *odd template*;

or contains either:

- G has a *universal vertex*; or
- G has a *clique cutset*.



v is a universal vertex



v attaches to a clique

Some complexity results on $\mathcal{C}_k$

The following results are proved by *Horsfield (2022)*.

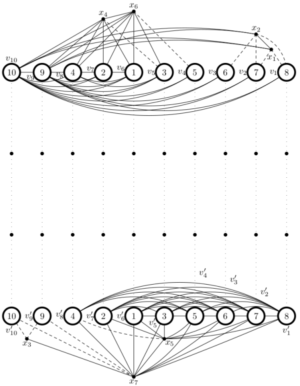
1. Given a graph G, **deciding whether** $G \in \mathcal{C}_k$ (for some odd $k \geq 7$) can be done in $\mathcal{O}(n^8)$, where $n = |V(G)|$.
2. There is an algorithm for the **Maximum Weight Clique problem** in $\mathcal{C}_k$ that works in $\mathcal{O}(n^2 m)$ where $n = |V(G)|$ and $m = |E(G)|$.
3. There is an algorithm for the **Maximum Weight Independent Set problem** in $\mathcal{C}_k$ that works in $\mathcal{O}(n^3 m)$ where $n = |V(G)|$ and $m = |E(G)|$.

Adapting the idea into *even-hole-free* graphs

Is it possible to generalize the structure theorem into graphs:

- allowing k-holes and $(k+2)$-holes?
- three odd holes?
- all odd holes (i.e., *even-hole-free*)?

Idea: “relaxing” the length of the paths connecting the top part and the bottom part in the blowup of odd-template.



D. Sintiari.
Width parameters on even-hole-free graphs (PhD Thesis).
École Normale Supérieure de Lyon, 2021.

J. Horsfield, M. Preissmann, C. Robin, N.L.D. Sintiari, N. Trotignon, K. Vŭsković.
When all holes have the same length.
arXiv:2203.11571, 2022.

J. Horsfield.
Structural characterisations of hereditary graph classes and algorithmic consequences (PhD Thesis).
The University of Leeds, 2022.

K. Vŭsković.
Even-hole-free graphs: a survey.
The University of Leeds, 2010.

Thank you for listening!